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Steady supersonic flow past a curved cone

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ABSTRACT

This paper studies the steady supersonic flow past a Lipschitz curved cone. Under the assumptions that the cone has an opening angle less than a critical value and has sufficiently small total variation of the tangent of the perturbation and that the Mach number of incoming flow is sufficiently large, the global weak solution is constructed via Glimm scheme for $1 < \gamma < 3$.

1. Introduction

In this paper, we are concerned with the problem of the steady potential supersonic flow past a Lipschitz curved cone. The flow and the cone have axis-symmetry. The surface of the cone is formed by the rotation of the graph $\{y = b(x),\ x > 0\}$, where the function $b(x)$ is a Lipschitz function satisfying the following (see Fig. 1):

(A1) $b(x) < 0$ for $x > 0$ and

$$b(x) = b_0 x, \quad x \in [0, t_0],$$

for some constants $b_0 < 0$ and $t_0 > 0$; moreover

$$b'_+(x) = \lim_{t \to x+0} \frac{b(t) - b(x)}{t - x} \in BV\big([0, +\infty)\big),$$

and $b'_+(x)$ equals to some negative constant for $x > t_*$ for some $t_* > t_0$.

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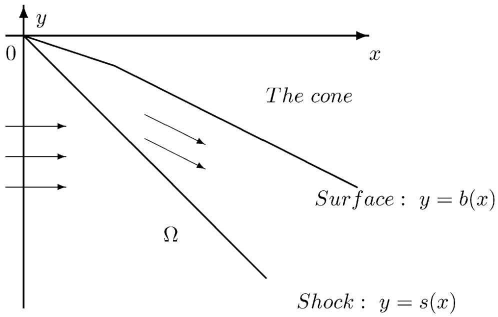

Fig. 1. Supersonic flow.

With the coordinates x and y, the equations of the flow can be written as

$$(\rho u)_x + (\rho v)_y = -\frac{\rho v}{y}, \tag{1.1}$$

$$v_x - u_y = 0, \tag{1.2}$$

with the Bernoulli equation:

$$\frac{u^2+v^2}{2} + \frac{c^2}{\gamma-1} = \frac{u_\infty^2}{2} + \frac{c_\infty^2}{\gamma-1}.$$

Here u and v are components of the flow velocity in the direction of the axis of the cone (or in the x-direction) and in the y-direction respectively; $(u_\infty, 0)$ is the velocity of the incoming flow; ρ is the density of the flow and c the sonic speed with $c = \sqrt{\gamma A \rho^{\gamma-1}}$ for some constant $A > 0$; $c_\infty = \sqrt{\gamma A \rho_\infty^{\gamma-1}}$ and ρ_∞ is the density of the incoming flow. Moreover we assume that:

(A2) The velocity and density of the incoming flow is supersonic, i.e.,

$$u_\infty > c_\infty = \sqrt{\gamma A \rho_\infty^{\gamma-1}}.$$

The purpose of this paper is to construct the global weak solution to (1.1)–(1.2) in Ω with

$$(u, v)|_{x \leqslant 0} = (u_\infty, 0) \tag{1.3}$$

and

$$(u, v) \cdot \vec{n}|_{\partial\Omega} = 0, \tag{1.4}$$

where

$$\Omega = \{(x, y) \mid x > 0,\ y < b(x)\}$$

and $\vec{n} = \vec{n}(x, y)$ is the normal to $\partial\Omega$ at differential points of b.

This problem has been studied by many authors, see, for instance, [7,8] and [16] and references therein. The simple case that the cone is formed by the rotation of a straight line is considered in [8], where the solution is given by the shooting method. The global weak solution is constructed via a modified Glimm scheme by Lien and Liu in [16], where they assume that the cone has a small opening angle and is a small perturbation of the surface of the straight cone, and that the incoming flow has a sufficiently large Mach number. By using different approach, Chen, Xin and Yin have constructed the piecewise smooth solution in [7], where they also assume that the cone has a small opening angle and the incoming flow has a sufficiently large Mach number but the cone surface is a small smooth perturbation of the straight one. Moreover, the solution in [7] has only one shock front issuing from the vertex of the cone. Different from the above cases, we consider the case that the cone has an opening angle less than a critical value. Therefore, the leading shock issuing from the vertex does not have small strength, while the smallness of the strength of the leading shock plays an important role in the analysis in [7] and [16]. Our analysis makes use of the modified Glimm scheme by Lien and Liu [16].

To overcome the difficulties caused by the large opening angle and to show the decreasing of the Glimm functional for the approximate solutions, we make full use of the following properties:

(1) The 2-waves disappear after their hitting the boundary. Meanwhile, a new 1-wave generated and its strength can be controlled by the disappeared 2-wave. This implies the decreasing of L_2^0 near the boundary.
(2) Centers of the approximate solution propagate away from the obstacle and toward the leading shock. When the wave with different centers on its both sides hitting the leading shock, one center disappears. This implies the decreasing of the L_c near the leading shock.

Here $L_2^0(J)$ and $L_c(J)$ denote the amount of all 2-waves crossing J and the total variation of the center function along the mesh curve J, for detail see Section 4. Moreover, to deal with the reflections between the leading shock and the obstacle, we prove the key estimate Proposition 4.3 for large u_∞, which is analogous to the case of wedge in [23] and implies the diminishing of weak waves after reflections. Then, combining the ideas from [16] and [22,23] with the above properties, we prove the decreasing of the modified Glimm functional, therefore get the convergence of the approximate solutions. For the Glimm scheme and its modification, see [1,9,10,16,19–21]. We also remark that related problems, e.g., supersonic flow past a wedge and multidimensional piston problems, etc., have been studied in [2–8,11,14,15,17,18].

The remaining of this paper is organized as follows. In Section 2, in a slightly different way from [7] and [16], we will give the asymptotic behaviour of eigenvalues and characteristic vectors, elementary waves of Riemann problem and self-similar solution of the unperturbed conical flow as u_∞ goes to infinity. Section 3 is devoted to the construction of approximate solution via Glimm scheme. The local interaction estimates are given in Section 4 for large u_∞. Finally in Section 5, we construct the Glimm functional and prove its monotonicity. The main theorem is also stated there.

2. Preliminary

In this section, we will give some quantitative analysis on the circular conical shock and the shock polar, elementary wave curves for homogeneous Euler system.

2.1. Auxiliary lemmas

Regarding x as the time variable, the system

$$(\rho u)_x + (\rho v)_y = 0, \tag{2.1}$$

$$v_x - u_y = 0 \tag{2.2}$$

is strictly hyperbolic with two distinct eigenvalues:

$$\lambda_1 = \frac{uv - c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}$$

and

$$\lambda_2 = \frac{uv + c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}$$

for $u > c_*$ and $u^2 + v^2 < q_*^2$. Here $c_* > 0$ and $q_* > 0$, which are critical sonic speed and critical speed given by the following:

$$c_*^2 = \frac{(\gamma - 1)u_\infty^2}{\gamma + 1} + \frac{2c_\infty^2}{\gamma + 1}$$

and

$$q_*^2 = u_\infty^2 + \frac{2c_\infty^2}{\gamma - 1}.$$

With $q = \sqrt{u^2 + v^2}$ and $\theta = \arctan \frac{v}{u}$, the eigenvalues can be rewritten as

$$\lambda_1 = \tan(\theta - \theta_{ma}), \tag{2.3}$$

$$\lambda_2 = \tan(\theta + \theta_{ma}) \tag{2.4}$$

where the Mach angle θ_{ma} is given by

$$\theta_{ma} = \arctan \frac{c}{\sqrt{q^2 - c^2}}.$$

By direct computation, we also have $\theta_{ma} = \arcsin \frac{c}{q} \in (0, \frac{\pi}{2})$.

Lemma 2.1. *For $u > c_*$ and $q < q_*$,*

$$\cos\big(\theta + (-1)^j \theta_{ma}\big) > 0, \quad j = 1, 2. \tag{2.5}$$

Therefore $\theta \pm \theta_{ma} \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Proof. Since

$$\big(u\sqrt{q^2 - c^2}\big)^2 - (vc)^2 = \big(u^2 - c^2\big)q^2 > 0$$

for $u > c_*$ and $q < q_*$, then

$$q^2 \cos\big(\theta + (-1)^j \theta_{ma}\big) = u\sqrt{q^2 - c^2} + (-1)^j vc > 0.$$

This leads to the result. The proof is complete. $\square$

Lemma 2.2. *For $u > c_*$ and $q < q_*$,*

$$\frac{\partial \theta}{\partial u} = -\frac{\sin\theta}{q}, \qquad \frac{\partial \theta}{\partial v} = \frac{\cos\theta}{q}, \tag{2.6}$$

$$\frac{\partial q}{\partial u} = \cos\theta, \qquad \frac{\partial q}{\partial v} = \sin\theta, \tag{2.7}$$

$$\frac{\partial \theta_{ma}}{\partial q} = -\frac{2c^2 + (\gamma-1)q^2}{2cq\sqrt{q^2-c^2}}, \tag{2.8}$$

$$\frac{\partial}{\partial q}\left(\frac{c}{\sqrt{q^2-c^2}}\right) = -\frac{q(2c^2+(\gamma-1)q^2)}{2cq\sqrt{(q^2-c^2)^3}}. \tag{2.9}$$

Lemma 2.3. *For $u > c_*$ and $q < q_*$,*

$$\frac{\partial \lambda_1}{\partial q} = \frac{2c^2 + (\gamma-1)q^2}{2cq\sqrt{q^2-c^2}} \sec^2(\theta - \theta_{ma}), \tag{2.10}$$

$$\frac{\partial \lambda_1}{\partial \theta} = \sec^2(\theta - \theta_{ma}). \tag{2.11}$$

By the computation above and using Lemma 2.2, we have the following:

Lemma 2.4. *For $u > c_*$ and $q < q_*$,*

$$(-\lambda_j, 1) \cdot \left(\frac{\partial \lambda_j}{\partial u}, \frac{\partial \lambda_j}{\partial v}\right) = g(q)\sec^3\bigl(\theta + (-1)^j\theta_{ma}\bigr), \tag{2.12}$$

where $j = 1, 2$ and

$$g(q) = \frac{2c^2+(\gamma-1)q^2}{2cq\sqrt{q^2-c^2}}\sin\theta_{ma} + \frac{\cos\theta_{ma}}{q} = \frac{\gamma+1}{2\sqrt{q^2-c^2}}.$$

This lemma implies that (2.1)–(2.2) is genuinely-nonlinear for $u > c_*$ and $q < q_*$. Then let

$$\tilde{r}_j(U) = \bigl(-\lambda_j(U), 1\bigr)^T$$

and

$$e_j(U) = \frac{1}{\tilde{r}_j(U) \cdot \nabla\lambda_j(U)},$$
$$r_j(U) = e_j(U)\tilde{r}_j(U),$$

where $j = 1, 2$.

Lemma 2.5. *For $u > c_*$ and $q < q_*$,*

$$e_j(U) = \frac{\cos^3(\theta + (-1)^j\theta_{ma})}{g(q)} > 0, \tag{2.13}$$

where $j = 1, 2$.

2.2. Circular conical flow

We consider the case that $b(x) = b_0 x$ for $x \in [0, +\infty)$. Due to [8], when b_0 is less than a critical value then the problem (1.1)–(1.4) has a self-similar solution $(u(\sigma), v(\sigma))$ with $\sigma = y/x$, and the solution consists of a conical shock front issuing from the vortex. Let $y = s_0 x$ be the location of shock front. Then, in this case the problem (1.1)–(1.4) becomes

$$\left(-\sigma^2\left(1-\frac{u^2}{c^2}\right)-\frac{uv}{c^2}\sigma\right)u_\sigma+\left(\frac{uv}{c^2}+\left(1-\frac{v^2}{c^2}\right)\sigma\right)v_\sigma+v=0, \quad s_0<\sigma<b_0, \tag{2.14}$$

$$u_\sigma+\sigma v_\sigma=0, \quad s_0<\sigma<b_0, \tag{2.15}$$

$$\rho(us_0-v)=\rho_\infty u_\infty s_0, \quad \sigma=s_0, \tag{2.16}$$

$$u+vs_0=u_\infty, \quad \sigma=s_0, \tag{2.17}$$

$$ub_0=v, \quad \sigma=b_0; \tag{2.18}$$

moreover,

$$\big(u(\sigma), v(\sigma)\big)=(u_\infty, 0), \quad \sigma<s_0. \tag{2.19}$$

Here s_0 is the slope of the shock front. With the Bernoulli equation, (2.14)–(2.15) can be rewritten in an equivalent form as

$$u_\sigma=\frac{vc^2}{(1+\sigma^2)c^2-(v-\sigma u)^2}, \tag{2.20}$$

$$v_\sigma=\frac{-vc^2}{\sigma((1+\sigma^2)c^2-(v-\sigma u)^2)}, \tag{2.21}$$

$$\rho_\sigma=\frac{\rho vc^2(v-\sigma u)}{\sigma((1+\sigma^2)c^2-(v-\sigma u)^2)}. \tag{2.22}$$

To study the self-similar solution, we need the following properties on the shock polar. Denote

$$b_*=\left(\frac{1}{2}\left(-1+\sqrt{\frac{\gamma+7}{\gamma-1}}\right)\right)^{1/2}. \tag{2.23}$$

Lemma 2.6. *For $1<\gamma<3$ and $b_0\in(-b_*, 0)$ and $\rho_\infty>0$, there exist constants $K'>0$, $K''>0$ and $K'''>0$, independent of u_∞, such that for $u_\infty>K'''$ the system of equations*:

$$\rho_+(u_+s_+-v_+)=\rho_\infty u_\infty s_+, \tag{2.24}$$

$$u_++v_+s_+=u_\infty, \tag{2.25}$$

$$u_+b_0-v_+=0, \tag{2.26}$$

$$\frac{u_+^2+v_+^2}{2}+\frac{c_+^2}{\gamma-1}=\frac{u_\infty^2}{2}+\frac{c_\infty^2}{\gamma-1} \tag{2.27}$$

has a unique solution (ρ_+, u_+, v_+, s_+) *with*

$$s_+ \in \left(b_0 - K' M_\infty^{-\frac{2}{\gamma-1}}, b_0\right).$$

Moreover,

$$s_+ \in \left(b_0 - K' M_\infty^{-\frac{2}{\gamma-1}}, b_0 - K'' M_\infty^{-\frac{2}{\gamma-1}}\right), \tag{2.28}$$

$$\lim_{u_\infty \to +\infty} \frac{u_+}{u_\infty} = \frac{1}{1+b_0^2}, \tag{2.29}$$

$$\lim_{u_\infty \to +\infty} \frac{v_+}{u_\infty} = \frac{b_0}{1+b_0^2}, \tag{2.30}$$

$$\lim_{u_\infty \to +\infty} \frac{u_+}{c_+} > 1, \tag{2.31}$$

where $M_\infty = \frac{u_\infty}{c_\infty}$.

Proof. Eqs. (2.25) and (2.26) give

$$u_+ = \frac{u_\infty}{1+b_0 s_+}, \tag{2.32}$$

$$v_+ = \frac{u_\infty b_0}{1+b_0 s_+}, \tag{2.33}$$

and with the help of (2.24), we obtain

$$\rho_+ = \frac{\rho_\infty (1+b_0 s_+) s_+}{s_+ - b_0}. \tag{2.34}$$

Then, substituting (2.32), (2.33) and (2.34) into Eq. (2.27), we have

$$\frac{1}{2}\left\{\frac{1+b_0^2}{(1+b_0 s_+)^2} - 1\right\} + \frac{1}{(\gamma-1)}\left\{\frac{(1+b_0 s_+)^{\gamma-1}|s_+|^{\gamma-1}}{M_\infty^2 |s_+ - b_0|^{\gamma-1}} - \frac{1}{M_\infty^2}\right\} = 0. \tag{2.35}$$

To solve Eq. (2.35), we set

$$f(s) = \frac{1}{2}\left\{\frac{1+b_0^2}{(1+b_0 s)^2} - 1\right\} + \frac{1}{\gamma-1}\left\{\frac{(1+b_0 s)^{\gamma-1}|s|^{\gamma-1}}{M_\infty^2 |s - b_0|^{\gamma-1}} - \frac{1}{M_\infty^2}\right\}$$

for $s < b_0$. Note that $b_0 < 0$. It is easy to verify that

$$\lim_{s \to b_0 - 0} f(s) = \lim_{s \to -\infty} f(s) = +\infty. \tag{2.36}$$

In addition, for $K > 0$,

$$f\left(b_0 - K M_\infty^{-\frac{2}{\gamma-1}}\right) = f_1\left(K, M_\infty^{-\frac{2}{\gamma-1}}\right),$$

where

$$f_1(K,t)=\frac{1}{2}\left\{\frac{1+b_0^2}{(1+b_0^2-b^0Kt)^2}-1\right\}+\frac{1}{\gamma-1}\left\{\frac{(1+b_0^2-b_0Kt)^{\gamma-1}|b_0-Kt|^{\gamma-1}}{K^{\gamma-1}}-t^{\gamma-1}\right\}.$$

Since

$$f_1(K,0)=\frac{1}{2}\left\{\frac{1}{1+b_0^2}-1\right\}+\frac{1}{\gamma-1}\left\{\frac{(1+b_0^2)^{\gamma-1}|b_0|^{\gamma-1}}{K^{\gamma-1}}\right\},$$

we can choose $K'>0$ and $K''>0$ so that

$$f_1(K',0)<0,\qquad f_1(K'',0)>0.$$

These lead to

$$f\big(b_0-K'M_\infty^{-\frac{2}{\gamma-1}}\big)<0$$

and

$$f\big(b_0-K''M_\infty^{-\frac{2}{\gamma-1}}\big)>0$$

for M_∞ sufficiently large, which implies that the equation $f(s)=0$ has two solutions which lie in $(b_0-K'M_\infty^{-\frac{2}{\gamma-1}},b_0-K''M_\infty^{-\frac{2}{\gamma-1}})$ and $(-\infty,b_0-K'M_\infty^{-\frac{2}{\gamma-1}})$ respectively.

On the other hand, the property of the shock polar implies that $f(s)=0$ has at most two solutions in $(-\infty,b_0)$. Therefore, $f(s)=0$ has a unique solution in $(b_0-K'M_\infty^{-\frac{2}{\gamma-1}},b_0-K''M_\infty^{-\frac{2}{\gamma-1}})$, which leads to the uniqueness of (ρ_+,u_+,v_+,s_+) and

$$s_+\in\big(b_0-K'M_\infty^{-\frac{2}{\gamma-1}},b_0-K''M_\infty^{-\frac{2}{\gamma-1}}\big).\tag{2.37}$$

Then, by (2.32)–(2.34) and (2.37), we can get the desired estimates. The proof is complete. □

Denote

$$f(s,b)=\frac{1}{2}\left\{\frac{1+b^2}{(1+bs)^2}-1\right\}+\frac{1}{\gamma-1}\left\{\frac{(1+bs)^{\gamma-1}|s|^{\gamma-1}}{M_\infty^2(b-s)^{\gamma-1}}-\frac{1}{M_\infty^2}\right\}$$

for $s<b<0$.

Lemma 2.7. *For any $s\in[5b_0,b_0]$, the equation $f(b,s)=0$ has a unique solution b_s with $b_s\in(s,0)$ for M_∞ sufficiently large. Moreover, there exists a constant K' depending only on b_0 such that*

$$|b_s-s|\leqslant K'M_\infty^{-\frac{2}{\gamma-1}}.$$

Proof. Differentiate the function f with respect to b, then

$$\frac{\partial f}{\partial b}=\frac{2(b-s)}{(1+bs)^3}-\frac{1}{M_\infty^2}\frac{(1+bs)^{\gamma-2}|s|^{\gamma-1}(s^2+1)}{(b-s)^\gamma}.$$

To estimate the number of the zero points of $\frac{\partial f}{\partial b}$, let

$$f_2(b)=b-s-\left\{\frac{1}{2M_\infty^2}(1+bs)^{\gamma-2}|s|^{\gamma-1}\left(s^2+1\right)(1+bs)^3\right\}^{\frac{1}{\gamma+1}}.$$

For M_∞ sufficiently large, $\frac{\partial f_2}{\partial b}>0$, therefore, $f_2(b)=0$ has at most one solution in the interval $(s,0)$. This implies that $\frac{\partial f}{\partial b}=0$ has at most one solution in the interval $(s,0)$. Therefore, the equation $f(b,s)=0$ has at most one solution b_s with $b_s\in(s,0)$ for M_∞ sufficiently large.

Now, by direct computation, we have

$$\lim_{b\to s+0} f(s,b)=+\infty$$

and

$$\lim_{M_\infty\to+\infty} f\left(s,s+KM_\infty^{-\frac{2}{\gamma-1}}\right)=\frac{1}{2}\left(\frac{1}{s^2+1}-1\right)+\frac{1}{\gamma-1}\frac{(1+s^2)^{\gamma-1}|s|^{\gamma-1}}{K^{\gamma-1}}<0$$

for suitable $K\gg 1$, which imply the existence of the solution b_s. Thus, the proof is complete. □

Denote $S_1((u_\infty,0))$ be the part of the shock polar corresponding to the λ_1 characteristic field, and let

$$S_1^-\left((u_\infty,0)\right)=\left\{(u,v)\in S_1\left((u_\infty,0)\right)\,\middle|\,c^2\leqslant u^2+v^2\leqslant u_\infty^2,\ v<0\right\}.$$

Due to [12,22] and [23], we can parameterize the shock polar $S_1^-((u_\infty,0))$ for Euler equations (2.1) and (2.2) through $(U_\infty,0)$ by a C^2-function $G: s\mapsto G(s,u_\infty)$, that is, $G(s,u_\infty)$ is the state that can be connected to u_∞ by a shock with the speed s and u_∞ is the left state. In addition, $G(s,u_\infty)$ is a supersonic state. We write $G(s,u_\infty)$ as $G(s)$ in the sequel for simplification and denote by $\tilde u(s)$ and $\tilde v(s)$ the components of $G(s)$, that is, $G(s)=(\tilde u(s),\tilde v(s))^T$. It is obvious that $G(s)$ solves the following equations:

$$\tilde\rho(\tilde u s-\tilde v)=\rho_\infty u_\infty s, \tag{2.38}$$

$$\tilde u+\tilde v s=u_\infty, \tag{2.39}$$

where $\tilde\rho$ is determined from

$$\frac{\tilde u^2+\tilde v^2}{2}+\frac{A\gamma\tilde\rho^{\gamma-1}}{\gamma-1}=\frac{u_\infty^2}{2}+\frac{c_\infty^2}{\gamma-1}. \tag{2.40}$$

Lemma 2.8. *For $s<\lambda_1(u_\infty,0)$, $\tilde v(s)/\tilde u(s)$ is a strictly monotonically increasing function of s.*

Proof. Denote $\theta(s) = \arctan \tilde{v}(s)/\tilde{u}(s)$. Due to [8] and [12], we know that there is at most one intersection of straight line $v = bu$ and $S_1^-((U_\infty, 0))$ for any $b < 0$, which is corresponding to the supersonic shock. This implies that $\theta(s)$ is strictly monotonic for s.

On the other hand, the properties of the shock polar tell us that $\theta(s) < \theta(\lambda(u_\infty))$ for s close to $\lambda(u_\infty)$ with $s < \lambda(u_\infty) < 0$ (see [8] and [12,22] and [23]). Therefore, $\theta(s)$ is a strictly monotonically increasing function of s. The proof is complete. $\square$

Now consider the conical flow. We recall some facts on the apple curves in [8]. Given a constant state (u_1^0, v_1^0) in the shock polar through the state $(u_\infty, 0)$. Let $(u_1(\sigma), v_1(\sigma))$ be the solution of Eqs. (2.14) and (2.15) with initial data

$$\big(u_1(s), v_1(s)\big) = \big(u_1^0, v_1^0\big)$$

and

$$s = \frac{u_\infty - u_1^0}{v_1^0}.$$

Then continue the solution $(u_1(\sigma), v_1(\sigma))$ till the end point $(u_1(\sigma_e), v_1(\sigma_e))$ so that $v_1(\sigma_e)/u_1(\sigma_e) = \sigma_e$. The collection of the end states $(u_1(\sigma_e), v_1(\sigma_e))$ forms an apple curve through $(u_\infty, 0)$. The solution $(u(\sigma), v(\sigma))$ to (2.14)–(2.18) can be found by the shooting method (see [8]). Therefore,

$$u(\sigma)\sigma - v(\sigma)|_{s_0<\sigma<b_0} \neq 0 \tag{2.41}$$

and

$$u(b_0)b_0 - v(b_0) = 0.$$

Indeed we have the following.

Lemma 2.9. *There holds the following*

$$u(s_0)s_0 - v(s_0) < 0.$$

Therefore,

$$u(\sigma)\sigma - v(\sigma) < 0$$

for $s_0 < \sigma < b_0$.

Proof. Since $s_0 < 0$, then the Rankine–Hugoniot conditions give

$$u(s_0)s_0 - v(s_0) = \frac{\rho_\infty u_\infty s_0}{\rho(s_0)} < 0.$$

Therefore, by (2.41), we have

$$u(\sigma)\sigma - v(\sigma) < 0$$

for $s_0 < \sigma < b_0$. The proof is complete. $\square$

Lemma 2.10. *For $u(s_0) > c_*$, there holds the following*

$$\sqrt{1+s_0^2}c(s_0) - \big(v(s_0) - s_0 u(s_0)\big) > 0, \tag{2.42}$$

where $c(s_0)$ is the sonic speed given by the Bernoulli equation.

Proof. Let $f_3(\tau) = (1+\tau^2)c^2(s_0) - (v(s_0) - \tau u(s_0))^2$. Then

$$f_3\big(\lambda_1\big(u(s_0), v(s_0)\big)\big) = f_3\big(\lambda_2\big(u(s_0), v(s_0)\big)\big) = 0.$$

Moreover, the Lax entropy conditions give that

$$\lambda_1\big(u(s_0), v(s_0)\big) < s_0 < \lambda_2\big(u(s_0), v(s_0)\big). \tag{2.43}$$

On the other hand, the coefficient of the term τ^2 in $f_3(\tau)$ is $(c(s_0))^2 - (u(s_0))^2$, which is negative since $u(s_0) > c_*$. Then, with (2.43), we have

$$f_3(s_0) > 0,$$

which proves the lemma by Lemma 2.9. $\square$

Lemma 2.11. *Suppose that $u(s_0) > c_*$. Then for $s_0 < \sigma < b_0$ there hold the following,*

$$\frac{\partial u}{\partial \sigma} < 0, \qquad \frac{\partial v}{\partial \sigma} < 0,$$

and

$$c(\sigma) - \frac{v(\sigma) - \sigma u(\sigma)}{\sqrt{1+\sigma^2}} > c(s_0) - \frac{v(s_0) - \sigma u(s_0)}{\sqrt{1+s_0^2}} > 0.$$

Proof. Let

$$\sigma_* = \sup\left\{\sigma_0 \,\middle|\, c(\sigma) - \frac{v(\sigma) - \sigma u(\sigma)}{\sqrt{1+\sigma^2}} > 0 \text{ and } v(\sigma) < 0 \text{ for } \sigma \in [s_0, \sigma_0]\right\}.$$

By Lemma 2.10, $\sigma_* > s_0$. We will show that $\sigma_* \geqslant b_0$. Assume, to reach a contradiction, that $\sigma_* < b_0$. Then,

$$v(\sigma_*)\left\{c(\sigma_*) - \frac{v(\sigma_*) - \sigma_* u(\sigma_*)}{\sqrt{1+\sigma_*^2}}\right\} = 0. \tag{2.44}$$

Moreover, by Lemma 2.9,

$$v - \sigma u > 0, \qquad c(\sigma) + \frac{v(\sigma) - \sigma u(\sigma)}{\sqrt{1+\sigma^2}} > 0$$

for $\sigma \in [s_0, \sigma_*]$. This leads to the following,

$$\big(1+\sigma^2\big)c^2(\sigma) - \big(v(\sigma) - \sigma u(\sigma)\big)^2 > 0$$

for $\sigma \in [s_0, \sigma_*]$. Then, by (2.20)–(2.22), we have

$$u_\sigma < 0, \qquad v_\sigma < 0, \qquad c_\sigma > 0$$

for $\sigma \in [s_0, \sigma_*]$, which indeed imply that

$$\left(\frac{v - \sigma u}{\sqrt{1+\sigma^2}}\right)_\sigma = \frac{v_\sigma - \sigma u_\sigma}{\sqrt{1+\sigma^2}} - \frac{u + \sigma v}{(1+\sigma^2)^{3/2}} < 0$$

for $\sigma \in [s_0, \sigma_*]$. Therefore,

$$c(\sigma_*) - \frac{v(\sigma_*) - \sigma_* u(\sigma_*)}{\sqrt{1+\sigma_*^2}} > c(s_0) - \frac{v(s_0) - \sigma u(s_0)}{\sqrt{1+s_0^2}} > 0$$

and

$$v(\sigma_*) < v(s_0) < 0,$$

which yield the contradiction to (2.44). Then it follows that $\sigma_* = b_0$. The proof is complete. $\square$

Lemma 2.12. *For $1 < \gamma < 3$ and $b_0 \in (-b_*, 0)$ and $\rho_\infty > 0$, there exist constants $K' > 0$, $K'' > 0$ and $K''' > 0$, independent of u_∞, such that for $u_\infty > K'''$, the problem (2.14)–(2.18) has a unique solution $(u(\sigma), v(\sigma), s_0)$ constituted by a supersonic conical shock front issuing from the vertex. Moreover,*

$$\lim_{u_\infty \to +\infty} \frac{u(\sigma)}{u_\infty} = \frac{1}{1+b_0^2} = \cos^2\theta_0, \tag{2.45}$$

$$\lim_{u_\infty \to +\infty} \frac{v(\sigma)}{u_\infty} = \frac{b_0}{1+b_0^2} = \sin\theta_0 \cos\theta_0, \tag{2.46}$$

$$s_0 = b_0 + O(1) u_\infty^{-\frac{2}{\gamma-1}}, \tag{2.47}$$

$$\frac{\rho}{\rho_\infty} = \left\{\frac{\gamma-1}{2} \frac{u_\infty^2 s_0^2}{c_\infty(1+s_0^2)}\right\}^{\frac{1}{\gamma-1}} \big(1 + O(1) u_\infty^{-2}\big), \tag{2.48}$$

and

$$\lim_{u_\infty \to +\infty} \left(\frac{u(\sigma)}{c(\sigma)}\right)^2 = \frac{2}{(\gamma-1) b_0^2 (1+b_0^2)} > 1, \tag{2.49}$$

$$\cos\big(\theta_0 \pm \theta_{ma}^0\big) > 0, \tag{2.50}$$

where $\theta_0 = \arctan b_0$ and $\theta_{ma}^0 = \lim_{u_\infty \to +\infty} \theta_{ma}$; $O(1)$ stands for a bounded quantity as $u_\infty \to +\infty$.

Proof. Lemma 2.6 implies that the straight line $v = bu$ intersects the shock polar through $(u_\infty, 0)$. Then, due to the structure of the apple curve given in [8], the problem (2.14)–(2.18) has a unique solution $(u(\sigma), v(\sigma))$ with $u(s_0) > c_*$. Now, we have to prove (2.47) at first. To this end, let

$$b_1 = \frac{v(s_0)}{u(s_0)}$$

and denote $u_1 = u(s_0)$ and $v_1 = v(s_0)$. By Lemma 2.11,

$$b_0 u(s_0) \leqslant b_0 u(b_0) = v(b_0) \leqslant v(s_0),$$

which leads to

$$b_0 \leqslant b_1 \leqslant 0.$$

This enables us to use Lemma 2.8 to get

$$s_+ < s_0 < b_0.$$

Therefore, by Lemma 2.6,

$$s_0 = b_0 + O(1) u_\infty^{-\frac{2}{\gamma-1}}.$$

On the other hand, in the same way as in proof of Lemma 2.6, we can prove

$$f(b_1, s_0) = 0.$$

Then, by Lemma 2.7,

$$b_1 = s_0 + O(1) u_\infty^{-\frac{2}{\gamma-1}} = b_0 + O(1) u_\infty^{-\frac{2}{\gamma-1}}.$$

Since $(u(s_0), v(s_0))$ solves the equations:

$$u(s_0) + v(s_0) s_0 = u_\infty,$$
$$u(s_0) b_0 - v(s_0) = 0,$$

then, with estimates on s_0 and b_1, we have

$$\frac{u(s_0)}{u_\infty} = \frac{1}{1 + b_1 s_0} = \frac{1}{1 + b_0^2} + O(1) u_\infty^{-\frac{2}{\gamma-1}},$$
$$\frac{v(s_0)}{u_\infty} = \frac{b_1}{1 + b_1 s_0} = \frac{b_0}{1 + b_0^2} + O(1) u_\infty^{-\frac{2}{\gamma-1}}.$$

Therefore, making use of the monotonicity of (u, v) again that

$$b_0 u(s_0) \leqslant b_0 u(\sigma) \leqslant b_0 u(b_0) = v(b_0) \leqslant v(\sigma) \leqslant v(s_0),$$

we can derive the estimates (2.45)–(2.46), from which follow the estimates (2.48) and (2.49).

Now, it suffices to prove (2.50). Since

$$\lim_{u_\infty\to+\infty}\frac{(u\sqrt{q^2-c^2})^2-(vc)^2}{u_\infty^4}=\lim_{u_\infty\to+\infty}\frac{(u^2/c^2-1)q^2c^2}{u_\infty^4}>0,$$

then

$$\begin{aligned}\cos^2\bigl(\theta_0\pm\theta_{ma}^0\bigr)&=\lim_{u_\infty\to+\infty}\cos^2(\theta\pm\theta_{ma})\\&=\lim_{u_\infty\to+\infty}\frac{(u\sqrt{q^2-c^2}\pm vc)/u_\infty^2}{q^2/u_\infty^2}>0.\end{aligned}$$

The proof is complete. $\square$

2.3. Estimates on the unperturbed shock front for (2.1) and (2.2)

We turn to consider the shock polar for the homogeneous Euler system (2.1) and (2.2).

Lemma 2.13. *There holds the following*:

$$\lim_{u_\infty\to+\infty}\frac{\tilde u(s_0)}{u_\infty}=\cos^2\theta_0, \tag{2.51}$$

$$\lim_{u_\infty\to+\infty}\frac{\tilde v(s_0)}{u_\infty}=\cos\theta_0\sin\theta_0, \tag{2.52}$$

and

$$\lim_{u_\infty\to+\infty}\frac{\tilde u_s(s_0)}{u_\infty}=-\sin2\theta_0\cos^2\theta_0, \tag{2.53}$$

$$\lim_{u_\infty\to+\infty}\frac{\tilde v_s(s_0)}{u_\infty}=\cos2\theta_0\cos^2\theta_0. \tag{2.54}$$

Here $\tilde u_s(s_0)=\frac{\partial\tilde u(s_0)}{\partial s}$ *and* $\tilde v_s(s_0)=\frac{\partial\tilde v(s_0)}{\partial s}$.

Proof. The first two equalities come from Lemma 2.12. To get the last two equalities, we take the derivative to the Rankine–Hugoniot equations (2.38)–(2.40) with respective to s, then

$$\left(-\frac{(\tilde us-\tilde v)\tilde u}{\tilde c^2}+s\right)\frac{\tilde u_s}{u_\infty}-\left(\frac{(\tilde us-\tilde v)\tilde v}{\tilde c^2}+1\right)\frac{\tilde v_s}{u_\infty}+\frac{\tilde u}{u_\infty}=\frac{\rho_\infty}{\tilde\rho}, \tag{2.55}$$

$$\frac{\tilde u_s}{u_\infty}+\frac{\tilde v_s}{u_\infty}+\frac{\tilde v}{u_\infty}=0. \tag{2.56}$$

Since by (2.51) and (2.52) we have

$$\lim_{u_\infty\to+\infty}\det\begin{pmatrix}-\frac{(\tilde us-\tilde v)\tilde u}{\tilde c^2}+s & \frac{(\tilde us-\tilde v)\tilde v}{\tilde c^2}+1\\ 1 & 1\end{pmatrix}\bigg|_{s=s_0}=\det\begin{pmatrix}b_0 & -1\\ 1 & 1\end{pmatrix}\neq0$$

for $b_0\in(-b_*,0)$, then the limits, $\lim_{u_\infty\to+\infty}\frac{\tilde u_s(s_0)}{u_\infty}$ and $\lim_{u_\infty\to+\infty}\frac{\tilde v_s(s_0)}{u_\infty}$, do exist. Now let $s=s_0$ and take the limits in (2.55) and (2.56), then by Lemma 2.12, we have

$$b_0 \lim_{u_\infty \to +\infty} \frac{\tilde{u}_s}{u_\infty} - \lim_{u_\infty \to +\infty} \frac{\tilde{v}_s}{u_\infty} + \frac{1}{1+b_0^2} = 0, \tag{2.57}$$

$$\lim_{u_\infty \to +\infty} \frac{\tilde{u}_s}{u_\infty} + b_0 \lim_{u_\infty \to +\infty} \frac{\tilde{v}_s}{u_\infty} + \frac{1}{1+b_0^2} = 0, \tag{2.58}$$

which give the last two formulas for $\lim_{u_\infty \to +\infty} \frac{\tilde{u}_s(s_0)}{u_\infty}$ and $\lim_{u_\infty \to +\infty} \frac{\tilde{v}_s(s_0)}{u_\infty}$ by Cramer's law. The proof is complete. □

2.4. Elementary wave curves for (2.1) and (2.2)

We consider the elementary wave curves for homogeneous Euler system (2.1) and (2.2) near the states in the circular conical flow. Denote by $W(b_0, u_\infty)$ the curve formed by the states on the conical flow got in Section 2.2, that is, $W(b_0, u_\infty)$ is the curve formed by the state $(u(\sigma), v(\sigma))$ which is the solution to (2.14)–(2.18). As in [22,23] (see also [10,13,20]), we parameterize the elementary j-wave curves for the system (2.1)–(2.2) in a neighborhood $O(W(b_0, u_\infty))$ of $W(b_0, u_\infty)$ by

$$\alpha_j \mapsto \Phi_j(\alpha_j, U_l)$$

with $\Phi_j \in C^2$ and

$$\frac{\partial \Phi_j}{\alpha_j}\bigg|_{\alpha_j=0} = r_j(U_l)$$

for $j = 1, 2$, where $U_l \in O(W(b_0, u_\infty))$; moreover, $\alpha_j > 0$ for $R_j^+(U_l)$ and $\alpha_j < 0$ for $S_j^-(U_l)$.

In the sequel, we denote $\Phi(\alpha_1, \alpha_2, U_l) = \Phi_2(\alpha_2, \Phi_1(\alpha_1, U_l))$ and use the notation $\{U_l, U_r\}$ to denote the solution of the equations $\Phi(\alpha_1, \alpha_2, U_l) = U_r$, that is, $\{U_l, U_r\} = (\alpha_1, \alpha_2)$. Then, [10] gives the following interaction estimates (see also [1,9,20] and [21]).

Lemma 2.14. *Let $U_l \in W(b_0, u_\infty)$, and suppose that α, β, γ satisfy*

$$\Phi\big(\alpha, \Phi(\beta, U_l)\big) = \Phi(\gamma, U_l).$$

Then,

$$\gamma = \alpha + \beta + O(1)\Delta(\alpha, \beta),$$

where $\Delta(\alpha, \beta) = \sum\{|\alpha_i||\beta_j|\colon \alpha_i$ and β_j approach$\}$, where the bound of $O(1)$ depends continuously on $u_\infty < +\infty$.

Lemma 2.15. *For $U_l \in W(b_0, u_\infty)$, there hold the following:*

$$\lim_{u_\infty \to +\infty} g(q)u_\infty > 0, \qquad \lim_{u_\infty \to +\infty} \frac{e_j(U_l)}{u_\infty} > 0, \quad j = 1, 2,$$

$$\lim_{u_\infty \to +\infty} (1/u_\infty) r_j(U_l) = \big(-\tan\big(\theta_0 + (-1)^j \theta_{ma}^0\big), 1\big)^T \lim_{u_\infty \to +\infty} \frac{e_j(U_l)}{u_\infty}.$$

Proof. Direct computation and Lemma 2.7 give

$$\lim_{u_\infty\to+\infty} g(q)u_\infty = \lim_{u_\infty\to+\infty} \frac{(\gamma+1)u_\infty}{2\sqrt{q^2-c^2}} > 0,$$

and

$$\lim_{u_\infty\to+\infty} \frac{e_j(U_l)}{u_\infty} = \cos^3\bigl(\theta_0 + (-1)^j\theta_{ma}^0\bigr) \lim_{u_\infty\to+\infty} \frac{1}{u_\infty g(q)} > 0. \qquad \square$$

Lemma 2.16. *For $U_l \in W(b_0, u_\infty)$,*

$$\lim_{u_\infty\to+\infty} \frac{\det(r_1(U_l), r_2(U_l))}{u_\infty^2} \neq 0,$$
$$\lim_{u_\infty\to+\infty} \frac{\det(r_2(G(s_0)), G'(s_0))}{u_\infty^2} \neq 0.$$

Proof. By Lemmas 2.1 and 2.11, we have

$$\lim_{u_\infty\to+\infty} \frac{\det(r_1(U_l), r_2(U_l))}{u_\infty^2} = \lim_{u_\infty\to+\infty} \left(\frac{e_1(U_l)e_2(U_l)}{u_\infty^2}\right) \frac{\sin(2\theta_{ma}^0)}{\cos(\theta_0-\theta_{ma}^0)\cos(\theta_0+\theta_{ma}^0)} \neq 0,$$

and noticing that $\theta_0 < 0$, by Lemmas 2.1, 2.8 and 2.11, we have

$$\lim_{u_\infty\to+\infty} \frac{\det(r_2(G(s_0)), G'(s_0))}{u_\infty^2} = \lim_{u_\infty\to+\infty} \left(\frac{e_2(U_l)}{u_\infty}\right)^2 \frac{\cos^2\theta_0 \sin(\theta_{ma}^0-\theta_0)}{\cos(\theta_0+\theta_{ma}^0)} \neq 0.$$

The proof is complete. $\square$

3. Approximate solutions

The approximate solutions for system (1.1)–(1.2) with (1.3)–(1.4) are constructed in the same way as in [16]. For $\Delta x > 0$, we choose a set of points $\{A_k\}_{k=0}$ with $A_k = (t_k, b(t_k))$ and $t_k = t_0 + k\Delta x$, and approximate the boundary by a set of line segments $\{\Gamma_k\}$ with $\Gamma_k = A_{k-1}A_k$. Denote $\Gamma_\Delta = \bigcup_{k\geqslant 1}\Gamma_k$.

Due to the hypotheses, the slope of Γ_k is negative and uniformly bounded. Then, we can extend Γ_k so that the extension of Γ_k and the x-axis intersect at the point $(X_k^*, 0)$ with

$$X_k^* = t_{k-1} - \frac{\Delta x b(t_{k-1})}{b(t_k) - b(t_{k-1})}.$$

By direct computation, we have

Lemma 3.1. *There holds the following*:

$$X_k^* - X_{k-1}^* = O(1)|\xi_k - \xi_{k-1}|b(t_{k-1})$$

for $k > 0$, where $O(1)$ is independent of k, and ξ_k is the slope of Γ_k, that is,

$$\xi_k = \frac{b(t_k) - b(t_{k-1})}{t_0}.$$

Remark 3.1. Due to (A1), $O(1)b(t_{k-1})$ in the last lemma can also be replaced by $O(1)$. In fact, to ensure that $O(1)b(t_{k-1})$ is uniformly bounded, we can also assume that $b(x)$ has some decay property.

We also denote that

$$\omega_0 = \omega(A_0) = \arctan \frac{b(t_0) - b(0)}{t_0},$$
$$\omega_k = \omega(A_k) = \arctan \xi_{k+1} - \arctan \xi_k, \quad k > 0,$$

and

$$\Omega_k = \{(x, y) \mid t_{k-1} \leqslant x < t_k,\ y \leqslant b_\Delta(x)\}, \qquad \Omega_\Delta = \bigcup_{k=1}^{\infty} \Omega_k,$$

where $b_\Delta(x) = b(t_{k-1}) + \xi_k(x - t_{k-1})$ for $x \in (t_{k-1}, t_k)$.

We now define the difference scheme. In the region $0 \leqslant x \leqslant t_0$, the approximate solution is the unperturbed conical flow centered at $(0,0)$. For $x = t_0$, the grid points are the intersection of $x = t_0$ with the self-similar rays centered at $(0,0)$,

$$y = (\xi_0 + h\Delta\sigma)x, \quad n = 0, -1, -2, \ldots.$$

Here $\Delta\sigma > 0$ is chosen so that the initial numerical grid on $x = t_0$ satisfies the usual C-F-L condition. We also choose an equi-distributed sequence $a_0, a_1, a_2, a_3, \ldots,$ in the interval $(0,1)$. The approximate solutions $U_{\Delta x,a}$ and the numerical grids are defined inductively in k, $x = t_k$, $k = 0, 1, 2, \ldots,$ as follows:

Suppose that the approximate solution has been defined for $x < t_k$ and the grid points have been defined for $x = t_l$, $l \leqslant k$. Let the grid points on $x = t_k$ be denoted by $y_0(k) > y_1(k) > y_2(k) > \cdots$, with $y_0(k) = b(t_0 + k\Delta x)$, and let

$$a_{k,n} = y_n(k) + a_k\big(y_{n+1}(k) - y_n(k)\big).$$

The approximate solution $U_{\Delta x,a}$ is a piecewise smooth solution of the self-similar system (2.14)–(2.15) on the region Ω_k for $l \leqslant k$. That is, at any continuous point (x, y) of this approximate solution, it has the form

$$U_{\Delta x,a}(x, y) = U_{self}\left(\frac{y}{x - X^*}\right),$$

where U_{self} is a solution of the self-similar system (2.14)–(2.15) and $X^* = X^*(x, y)$ is a piecewise constant and right continuous function. We call X^* the center as in [16]. As part of induction hypotheses, we also assume that the center X^* has been defined for $x < t_k$ and that $X^* \in \{X_j^*,\ j \geqslant 0\}$ for $x < t_k$.

Then, define

$$X^*(t_k, y) = X^*(t_k - 0, a_{k,n}), \quad y_n(k) < y < y_{n+1}(k).$$

Now we define the approximate solution in Ω_{k+1}. For $y_n(k) < y < y_{n+1}(k)$, $U_{\Delta x,a}(t_k, y)$ is defined to be the solution of (2.14)–(2.15) with the self-similar variable

$$\sigma = \frac{y}{t_k - X^*(t_k, y)}$$

and the initial data

$$U_{\Delta x,a}(t_k, a_{k,n}) = U_{\Delta x,a}(t_k - 0, a_{k,n}), \quad n = 0, -1, -2, \ldots.$$

The discontinuities at the grid point $(t_k, y_n(k))$, $n = -1, -2, \ldots,$ are resolved by solving the Riemann problem for (2.1)–(2.2) with initial data

$$U|_{x=t_k} = \begin{cases} U_{\Delta x,a}(t_k, y_n(k) - 0) & \text{if } y < y_n(k), \\ U_{\Delta x,a}(t_k, y_n(k) + 0) & \text{if } y > y_n(k). \end{cases}$$

The solution of this Riemann problem can be written as $U((y - y_n(k))/(x - t_k))$ and consists of rarefaction waves and shock waves. Let the lower edge of the 1-wave of the Riemann solution at $(t_0 + k\Delta x, y_n(k))$ be $(y - y_n(k))/(x - t_k) = \xi_{k,n}$. Then, define the center at (x, y) by

$$X^*(x, y) = X^*_{k,n}, \quad \text{for } (x, y) \in \Omega_{k+1,n},$$

where $X^*_{k,n} = X^*(t_k - 0, a_{k,n})$ and

$$\Omega_{k+1,n} = \left\{(x, y) \;\middle|\; t_k \leqslant x < t_{k+1},\ \frac{y - y_n(k)}{x - t_k} < \xi_{k,n} \text{ and } \frac{y - y_{n-1}(k)}{x - t_k} > \xi_{k,n-1}\right\}.$$

As in Lien and Liu [16], the numerical grids on $x = t_{k+1}$ are defined to be on the self-similar rays through the grids on $x = t_k$. Let

$$\sigma = \sigma(x, y) = y/\big(x - X^*(x, y)\big).$$

Due to the above steps σ is well defined. Denote

$$\sigma_{n-1/2}(k) = \sigma\left(t_k - 0, \frac{y_{n-1}(k) + y_n(k)}{2}\right), \quad n \leqslant 0,\ k \geqslant 0.$$

Then, as in [16], the approximate solution $U_{\Delta x,a}$ in $t_k < x < t_{k+1}$, $\sigma_{n-1/2}(k) < \sigma(x, y) < \sigma_{n+1/2}(k)$ is defined according to (2.14)–(2.15) along the ray $(y - y_n(k))/(x - t_k) = \xi$ with the initial data $U(\xi)$ at $x = t_k$, where $\sigma = y/(x - X^*_{k,n})$ for $\xi > \xi_{k,n}$ and $\sigma = y/(x - X^*_{k,n-1})$ for $\xi < \xi_{k,n}$; $U(\xi)$ is given by the solution of the above Riemann problem.

On the obstacle, $(y - y_0(k))/(x - t_k) = \xi_{k+1}$, a 1-shock (or 1-rarefaction) wave emerges when the obstacle changes angle toward (or away from) the flow. For this, we solve the initial–boundary problem for (2.1)–(2.2) with the initial data:

$$U(t_k, y) = U_{\Delta x,a}\big(t_k - 0, y_0(k) - 0\big), \quad y_{-1}(k) < y < y_0(k),$$

and with a boundary condition posed at Γ_k:

$$v/u = \xi_{k+1}.$$

Then, the approximate solution is extended to $\{(x, y) \mid t_k \leqslant x < t_{k+1},\ y_0(k) + 1/2(y_0(k) - y_{-1}(k)) < y < y_0(k) + (x - t_k)\xi_{k+1}\}$ as before with $\sigma = y/(x - X^*_{k+1})$.

The leading strong conical shock front next to the uniform upstream flow is traced continuously. Suppose that the approximate solution is constructed for $x < t_k$. Let $(x, y_f(x))$ denote the locus of the front of the leading strong 1-shock. Suppose that $y_{j_f-1}(k) < y_f(t_k) < y_{j_f}(k)$. As in [16], the interval

$y_{j_f-1}(k) < y < y_{j_f+1}(k)$ is also called the front region at $x = t_k$. Inside the front region, we first solve the self-similar solution to (2.14)–(2.15) with the initial data

$$U(t_k, a_{k,j_f}) = U_{\Delta x,a}(t_k - 0, a_{k,j_f})$$

with the self-similar variable

$$\sigma = \frac{y}{x - X^*(t_k, a_{k,j_f})}.$$

Denote the solution $U_{self}(t_k, y)$. Next, we solve the Riemann problem (2.1)–(2.2) so that

$$U(t_k, y) = \begin{cases} U_\infty, & \text{for } y < y_f(t_k), \\ U_{self}(t_k, y_f(k) + 0), & \text{for } y_f(t_k) < y < y_{j_f+1}(k). \end{cases}$$

The solution $U(x, y)$ contains a weak 2-wave and a relatively strong 1-shock wave, $\{U_\infty, U_+\}$, with speed s_{k+1}. Solve again Eqs. (2.14)–(2.15) in the interval $y_f(t_k) < y < y_{j_f+1}(k)$ with the initial value $U(t_k, y_f(t_k) + 0) = U_+$ and self-similar variable $\sigma = \sigma(x, y) = \frac{y}{x - X^*(t_k, a_{k,j_f})}$. Denote the solution by $U_-(\sigma)$. Now we can define the approximate solution in the front region as follows:

$$U_{\Delta x,\theta}(x, y) = \begin{cases} U_\infty, & \text{for } y < y_f(t_k), \\ U_-(\sigma(x, y)), & \text{for } y_f(t_k) < y < y_{j_f+1}(k). \end{cases}$$

The discontinuities at the point $(t_k, y_{j_f+1}(k))$ are resolved by the same construction as before. Moreover, define the center near the leading shock as

$$X^*(x, y) = X^*_{k,j_f}, \quad \text{for } t_k < x < t_{k+1}, \quad s_{k+1} < \frac{y - y_{j_f+1}}{x - t_k} < \xi_{k,j_f+1}.$$

4. Local interaction estimates

In this section, we consider various interactions and establish the estimates. Firstly, by direct computations, we give the following lemma.

Lemma 4.1. *For any C^2-function f, there holds the following*:

$$f(x, y) - f(x, 0) - f(0, y) + f(0, 0) = xy \int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x \partial y}(\lambda x, \mu y)\, d\lambda\, d\mu.$$

Consider the space-like curve consisting of line segments joining the random points one-by-one in turn. We call the unbounded piecewise linear space-like curve by mesh curve as in [20] (see also [10] and [16]). Every mesh curve I divides the region $\Omega_\Delta(a)$ into two parts: I^- and I^+, where I^- is the one containing the line $x = t_0$, and I^+ is the remaining part. Suppose that I and I' are two mesh curves, we denote $I' > I$ if every point of the mesh curve I' is either on I or contained in I^+. We call I' is an immediate successor of I if $I' > I$ and every mesh point of I' except one is on I.

For any pair of mesh curves I and J with $J > I$, let Λ be the region between I and J. We need only to study some special cases. According to the location of Λ, we divide the discussion into three cases as in [16]: (1) Λ is a diamond located between the shock and the cone; (2) Λ is a diamond covering a part of the surface of the cone; (3) Λ consists of three diamonds covering a part of the leading shock. They are discussed in the following three subsections respectively.

4.1. Interactions between weak waves

All waves except the leading shock are regarded as the weak waves. We want to get the interaction estimates for weak waves. To this end, let $h = h(\sigma - \sigma_0, \sigma_0, U_l)$ be the solution to the ODE system (2.14)–(2.15) with initial data

$$h|_{\sigma=\sigma_0} = U_l$$

for $U_l \in O(W(b_0, u_\infty))$. As in the above section, h is the basic solver for building block.

Lemma 4.2. *For $U_l \in W(b_0, u_\infty)$,*

$$\lim_{u_\infty \to +\infty} \frac{1}{u_\infty} \frac{\partial h}{\partial \Delta\sigma}\bigg|_{\Delta\sigma=0} = \left(\sin\theta_0 \cos^3\theta_0, -\cos^4\theta_0\right)^T,$$

where $\Delta\sigma = \sigma - \sigma_0$ and the superscript T denotes the transpose.

Proof. By (2.14)–(2.15),

$$\frac{\partial h}{\partial \Delta\sigma}\bigg|_{\Delta\sigma=0} = \left(\frac{v}{\sigma^2 + 1 - [(u\sigma - v)^2/c^2]}, \frac{-v}{\sigma\{\sigma^2 + 1 - [(u\sigma - v)^2/c^2]\}}\right)^T.$$

Then, we can get the desired result by Lemma 2.12. The proof is complete. □

Moreover, the direct computation shows that

Lemma 4.3. *Let $\sigma_k = y_k/(x_k - X)$ and $\bar\sigma_k = y_k/(x_k - \overline{X})$ for $k = 1, 2$ and centers $X > 0$ and $\overline{X} > 0$. Then,*

$$\Delta\bar\sigma = \Delta\sigma + O(1)|X - \overline{X}||\Delta\sigma|$$

with $\Delta\sigma = \sigma_2 - \sigma_1$ and $\Delta\bar\sigma = \bar\sigma_2 - \bar\sigma_1$.

Now we consider the interactions between the weak waves. Here and in the sequel, by weak waves we mean the waves except the leading shock. We also use the Greeks α, β, γ and so on to denote the weak waves and use the w_k to denote the solutions to (2.14)–(2.15). Suppose that α and β are waves entering Λ and that γ is the wave leaving Λ, with

$$\begin{aligned}
\left\{w_3\left(\sigma_2'\right), w_1(\sigma_2)\right\} &= \delta, \\
\left\{w_2(\bar\sigma_1), w_1(\sigma_1)\right\} &= \alpha, \\
\left\{w_3\left(\sigma_2'\right), w_2(\bar\sigma_2)\right\} &= \beta,
\end{aligned}$$

that is,

$$\begin{aligned}
\Phi\left(\delta, w_3\left(\sigma_2'\right)\right) &= w_1(\sigma_2), \\
\Phi\left(\alpha, w_2(\bar\sigma_1)\right) &= w_1(\sigma_1), \\
\Phi\left(\beta, w_3\left(\sigma_2'\right)\right) &= w_2(\bar\sigma_2),
\end{aligned}$$

where $\bar\sigma$, σ' and σ are computed in terms of the centers $\overline{X}$, X' and X, respectively. Then,

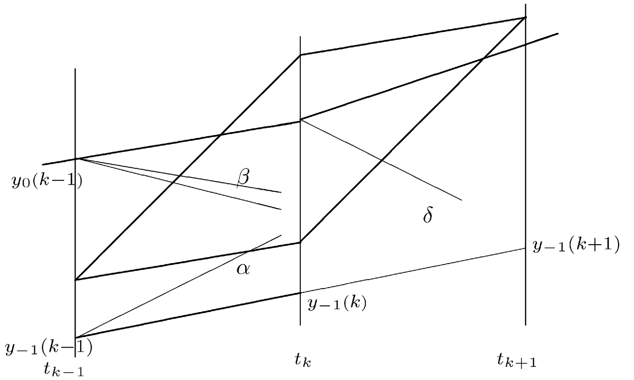

Fig. 2. Reflection at boundary.

Proposition 4.1. *There holds the following*:

$$\delta = \alpha + \beta + O(1)\big(\Delta(\alpha, \beta) + |\Delta\sigma||\alpha| + |\Delta\sigma||x_0|\big), \tag{4.1}$$

where $O(1)$ is independent of α, β, $\Delta\sigma$, x_0 but depends continuously on u_∞; x_0 denotes the difference of $\overline{X}$ and X, that is $x_0 = \overline{X} - X$; $\Delta\sigma = \sigma_2 - \sigma_1$.

For its proof, see Lien and Liu's paper [16].

4.2. Reflections at the boundary

We consider the wave interactions at the approximate boundary. Let $\varLambda$ be the diamond centered at $(t_k, y_0(k))$. The mesh curve and the "diamond" for this case have been shown in Fig. 2. Let β and δ be the 1-waves issuing from $(t_{k-1}, y_0(k-1))$ and $(t_k, y_0(k))$, respectively, with $w_2(\sigma_0)$ as the left state of δ. Let α be the 2-wave issuing from $(t_{k-1}, y_1(k-1))$ with $\{w_2(\sigma_1), w_1(\bar{\sigma}_1)\} = (0, \alpha)$. Denote U_l as the left state of α.

Then,

$$\varPhi_1\big(\delta, h(\sigma_0 - \sigma_1, \sigma_1, U_l)\big) \cdot \vec{n}_k = 0,$$
$$\varPhi_1\big(\beta, h\big(\bar{\sigma}_0 - \bar{\sigma}_1, \bar{\sigma}_1, \varPhi_2(\alpha, U_l)\big)\big) \cdot \vec{n}_{k-1} = 0,$$

where $\vec{n}_k = (-\sin\theta_k, \cos\theta_k)$ for $k \geqslant 0$, $\bar{\sigma}_i$ and σ_i are computed in terms of the centers $\overline{X}$ and X. Therefore,

$$\varPhi_1\big(\delta, h(\sigma_0 - \sigma_1, \sigma_1, U_l)\big) \cdot \vec{n}_k = \varPhi_1\big(\beta, h\big(\bar{\sigma}_0 - \bar{\sigma}_1, \bar{\sigma}_1, \varPhi_2(\alpha, U_l)\big)\big) \cdot \vec{n}_{k-1}. \tag{4.2}$$

To solve (4.2), we denote $\Delta\sigma = \sigma_0 - \sigma_1$, $\Delta\bar{\sigma} = \bar{\sigma}_0 - \bar{\sigma}_1$, $x_0 = X - \overline{X}$, and need the following.

Lemma 4.4. *There holds the following*:

$$\lim_{u_\infty \to +\infty} \frac{r_1(U_l) \cdot \vec{n}_0}{u_\infty} \neq 0. \tag{4.3}$$

Proof. The desired result comes from Lemma 2.11 and the following:

$$\left(-\tan\left(\theta_0-\theta_{ma}^0\right),1\right)\cdot(-\sin\theta_0,\cos\theta_0)=\frac{\cos\theta_{ma}^0}{\cos(\theta_0-\theta_{ma}^0)}\neq 0.$$

The proof is complete. □

Now we have the following estimates on δ.

Proposition 4.2. *Eq.* (4.2) *has a unique solution* $\delta=\delta(\beta,\alpha,\Delta\sigma,\Delta\bar{\sigma},x_0,\omega,U_l)$ *in a neighborhood of* $\delta=\beta=\alpha=\Delta\sigma=\Delta\bar{\sigma}=x_0=\omega=0$ *and* $U_l=U_\infty$, *with* $\delta\in C^2$. *Moreover,*

$$\delta=\beta+K_R\alpha+K_A\omega+O(1)u_\infty|\Delta\sigma||x_0| \tag{4.4}$$

with

$$K_R|_{\beta=\alpha=\omega=\Delta\sigma=x_0=0,\,\theta_k=\theta_0}=\frac{\cos^2(\theta_0+\theta_{ma})}{\cos^2(\theta_0-\theta_{ma})}, \tag{4.5}$$

and

$$\sup_{c_\infty<u_\infty<+\infty}|K_A|<+\infty, \tag{4.6}$$

where the bound of $O(1)$ *depends continuously on* u_∞.

Proof. Since

$$\frac{\partial}{\partial\delta}\Phi_1\left(\delta,h(\sigma_0-\sigma_1,\sigma_1,U_l)\right)\cdot\vec{n}_k\bigg|_{\delta=\beta=\alpha=\Delta\sigma=\Delta\bar{\sigma}=x_0=0,\,\omega,\,U_l=U_\infty}=r_1(U_l)\cdot\vec{n}_0,$$

then by Lemma 4.4 and by the implicit function theorem we can find a unique C^2 solution $\delta=\delta(\beta,\alpha,\Delta\sigma,\Delta\bar{\sigma},x_0,\omega,U_l)$ to (4.2) near $\delta=\beta=\alpha=\Delta\sigma=\Delta\bar{\sigma}=x_0=\omega=0$ and $U_l=U_\infty$.

Now we establish the estimates on δ. First, we have

$$\delta=O(1)|\Delta\bar{\sigma}-\Delta\sigma|+\delta_1,$$

where $\delta_1=\delta(\beta,\alpha,\Delta\sigma,\Delta\sigma,x_0,\omega,U_l)$ solves

$$\Phi_1\left(\delta_1,h(\sigma_0-\sigma_1,\sigma_1,U_l)\right)\cdot\vec{n}_k=\Phi\left(\beta,h\left(\sigma_0-\sigma_1,\bar{\sigma}_1,\Phi_2(\alpha,U_l)\right)\right)\cdot\vec{n}_{k-1}. \tag{4.7}$$

Let

$$\delta_2=\delta_1|_{\omega=0}=\delta(\beta,\alpha,\Delta\sigma,\Delta\sigma,x_0,0,U_l).$$

Then

$$\delta_1-\delta_2=K_A\omega$$

for some $K_A\in C^1$.

To estimate K_A, we compute $\frac{\partial\delta_1}{\partial\omega}|_{\omega=\alpha=\beta=x_0=0}$. To do this, $\frac{\partial}{\partial\omega}$(4.7) and let $\alpha=\beta=x_0=0$ in (4.7), then

$$r_1(\widetilde{U}_l)\cdot\vec{n}_{k-1}\frac{\partial\delta_1}{\partial\omega}\Big|_{\omega=\alpha=\beta=x_0=0}=\widetilde{U}_l\cdot\big(\cos(\omega+\tilde{\omega}_{k-1}),\sin(\omega+\tilde{\omega}_{k-1})\big),$$

where $\widetilde{U}_l=h(\Delta\sigma,\sigma_1,U_l)$. Therefore, Lemma 4.4 implies that the coefficient

$$K_A|_{\omega=\alpha=\beta=x_0=\Delta\sigma=0}=\frac{\partial\delta_1}{\partial\omega}\Big|_{\omega=\alpha=\beta=x_0=\Delta\sigma=0}$$

is uniformly bounded as $u_\infty\to+\infty$.

Now, denote $\delta_3(\beta,\Delta\sigma,x_0)=\delta_2|_{\alpha=0}$. Then, $\delta_3\in C^2$ and

$$\delta_2-\delta_3=K_R\alpha$$

for some $K_R\in C^1$.

To estimate K_R, we compute $\frac{\partial\delta_2}{\partial\alpha}|_{\alpha=\beta=x_0=\Delta\sigma=0}$. Note that δ_2 solves

$$\Phi_1\big(\delta_2,h(\sigma_0-\sigma_1,\sigma_1,U_l)\big)\cdot\vec{n}_{k-1}=\Phi\Big(\beta,h\big(\sigma_0-\sigma_1,\bar{\sigma}_1,\Phi_2(\alpha,U_l)\big)\Big)\cdot\vec{n}_{k-1}. \tag{4.8}$$

$\frac{\partial}{\partial\alpha}$(4.8) and let $\alpha=\beta=x_0=\Delta\sigma=0$, then

$$r_1(U_l)\cdot\vec{n}_{k-1}\frac{\partial\delta_2}{\partial\alpha}\Big|_{\alpha=\beta=x_0=\Delta\sigma=0}=r_2(U_l)\cdot\vec{n}_{k-1}.$$

Hence

$$K_R|_{\alpha=\beta=x_0=\Delta\sigma=0}=\frac{\partial\delta_2}{\partial\alpha}\Big|_{\alpha=\beta=x_0=\Delta\sigma=0}=\frac{r_2(U_l)\cdot\vec{n}_{k-1}}{r_1(U_l)\cdot\vec{n}_{k-1}}, \tag{4.9}$$

which gives the formula for K_R.

To finish the proof, we have to get the estimates on δ_3. Indeed, $\delta_3=\delta_3(\beta,\Delta\sigma,x_0)$ satisfies

$$\Phi_1\big(\delta_3,h(\sigma_0-\sigma_1,\sigma_1,U_l)\big)\cdot\vec{n}_{k-1}=\Phi\big(\beta,h(\sigma_0-\sigma_1,\bar{\sigma}_1,U_l)\big)\cdot\vec{n}_{k-1}. \tag{4.10}$$

Moreover, the uniqueness of solution for implicit function leads to

$$\delta_3(\beta,\Delta\sigma,0)=\delta_3(\beta,0,x_0)=\beta,$$

which implies

$$\delta_3(\beta,\Delta\sigma,x_0)-\delta_3(\beta,\Delta\sigma,0)-\delta_3(\beta,0,x_0)+\delta_3(\beta,0,0)=\widetilde{K}|\Delta\sigma||x_0|.$$

Thus,

$$\delta_3=\beta+\widetilde{K}|\Delta\sigma||x_0|. \tag{4.11}$$

Therefore, with the estimates on δ_j, $j=1,2,3$, we complete the proof. $\square$

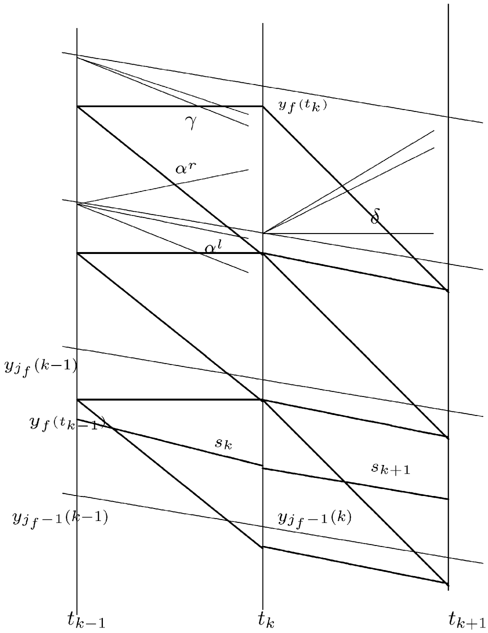

Fig. 3. Glimm scheme.

4.3. *Interactions at the leading shock front*

In this case, we will take three "diamond"s simultaneously. As shown in Fig. 3, let Δ_{k,j_f-1}, Δ_{k,j_f} and Δ_{k,j_f+1} be the diamonds centered in $(t_k, y_{j_f-1}(k)), (t_k, y_{j_f}(k))$ and $(t_k, y_{j_f+1}(k))$ respectively, and let $\Lambda = \Delta_{k,j_f-1} \cup \Delta_{k,j_f} \cup \Delta_{k,j_f+1}$. Denote α, γ the waves issuing from $(t_{k-1}, y_{j_{f+1}}(k-1))$ and $(t_{k-1}, y_{j_{f+2}}(k-1))$ respectively and entering Λ. Meanwhile, we divide α into parts: α^l and α^r with the waves α^l entering the "diamond" Δ_{k,j_f} and α^r entering Δ_{k,j_f+1}. We define $\delta = \{w_\delta^-, w_\delta^+\}$, which issues from $(t_k, y_{j_{f+1}}(k))$, that is, the Riemann solution generating the wave δ has left state w_δ^- and right state w_δ^+ in the initial data.

Remark 4.1. Note that $\arctan(v(\sigma)/u(\sigma)) \to b_0$ as $u_\infty \to +\infty$ for the background solution $(u(\sigma), v(\sigma))$, then in the sequel we can assume that $u_\infty \gg 1$ so that $\alpha^l = (\alpha_1^l, 0)$ and $\gamma = (\gamma_1, 0)$, that is, we assume that there is no 2-wave entering the diamond Δ_{k,j_f} and that there is no 2-wave issuing from $(t_{k-1}, y_{j_{f+2}}(k-1))$ and entering Δ_{k,j_f+1}.

We specify the center in this area as follows. The center in the region between the upper s_k and the lower edge of α is denoted as O_1, while the center in the region between the upper edge of α and the lower edge of γ is O_2, and the center above the lower edge of γ is O_3. Denote the x-coordinate of O_j by x_j^* $(j = 1, 2, 3)$, and denote $x_0 = |O_1 O_2| = |x_1^* - x_2^*|$ and $x_1 = |O_2 O_3| = |x_2^* - x_3^*|$. We also use the coordinates $\sigma = \sigma(x, y) = \frac{y}{x - x_1^*}$ and $\bar\sigma = \bar\sigma(x, y) = \frac{y}{x - x_2^*}$, $\sigma' = \sigma'(x, y) = \frac{y}{x - x_3^*}$. Then, denote $\sigma_\alpha = \sigma_{j_f+1}(k-1) = \sigma(t_{k-1}, y_{j_{f+1}}(k-1))$, $\bar\sigma_\alpha = \bar\sigma_{j_f+1}(k-1) = \bar\sigma(t_{k-1}, y_{j_{f+1}}(k-1))$, $\Delta\sigma_\alpha = \sigma_\alpha - \sigma_f(k-1)$, $\Delta\sigma_\gamma = \bar\sigma(t_k, y_{j_{f+1}}(k)) - \bar\sigma(t_{k-1}, y_{j_{f+2}}(k-1))$ and $\Delta\sigma_{s_k} = \sigma_f(k) - \sigma_f(k-1)$. Here $\sigma_f(k)$ and $\sigma_f(k-1)$ are the σ-coordinates of the points where the leading shock fronts s_{k+1} and s_k respectively issue.

Now, to get the estimates on δ, we consider the following,

$$\Phi\big(\alpha^l, h\big(\sigma_\alpha - \sigma_f(k-1), \sigma_f(k-1), G(s_k)\big)\big) = h\big(\bar\sigma_\alpha - \bar\sigma_f(k), \bar\sigma_f(k), \Phi_2\big(\epsilon, G(s_{k+1})\big)\big), \quad (4.12)$$

which gives the solution $U(x, y)$ as defined in Section 3. With the solution (s_{k+1}, ϵ) of (4.12), we can get the wave δ, which gives the approximate solution $U_{\Delta x,\theta}$ in $\{y_f(t_k) < y < y_{j_f}(k)\}$, see Section 3.

Proposition 4.3. *There hold the following*:

$$\delta_1 = \alpha_1^r + \gamma_1 + O(1)Q(\Lambda), \tag{4.13}$$

$$\delta_2 = \mu_w \Delta\sigma_{s_k} + K_w \alpha_1^l + \alpha_2^r + O(1)Q(\Lambda), \tag{4.14}$$

$$s_{k+1} = s_k + \mu_s \Delta\sigma_{s_k} + K_s \alpha_1^l + O(1)x_0|\Delta\sigma_\alpha| + O(1)x_0|\Delta\sigma_{s_k}|, \tag{4.15}$$

where $Q(\Lambda) = \Delta(\alpha^r, \gamma) + |\Delta\sigma_\alpha|(|\alpha^l| + |x_0| + |\Delta\sigma_{s_k}|) + |\Delta\sigma_\gamma|(x_1 + |\gamma|)$. *Moreover, when* $\alpha^l = \Delta\sigma_\alpha = \Delta\sigma_{s_k} = 0$ *and when* $s_k = s_0$, *we have*

$$K_w = \frac{\det(r_1(G(s_0)), G_s'(s_0))}{\det(r_2(G(s_0)), G_s'(s_0))}, \tag{4.16}$$

$$\mu_w = \frac{\det(\frac{\partial h}{\partial \Delta\sigma_{s_k}}, G_s'(s_0))}{\det(r_2(G(s_0)), G_s'(s_0))}, \tag{4.17}$$

$$K_s = \frac{\det(r_2(G(s_0)), r_1(G(s_0)))}{\det(r_2(G(s_0)), G_s'(s_0))}, \tag{4.18}$$

$$\mu_s = \frac{\det(r_2(G(s_0)), \frac{\partial h}{\partial \Delta\sigma_{s_k}})}{\det(r_2(G(s_0)), G_s'(s_0))}, \tag{4.19}$$

and

$$\lim_{u_\infty \to +\infty} \mu_s \in (-1, 0), \tag{4.20}$$

$$\lim_{u_\infty \to +\infty} K_s > 0, \tag{4.21}$$

where G_s' *denotes the derivative of* G.

Proof. We first consider Eq. (4.12). By Lemma 2.16 and the implicit function theorem, (4.12) has a unique solution (s_{k+1}, ϵ), which is C^2-function of the arguments, $\bar\sigma_\alpha - \bar\sigma_f(k)$, $\bar\sigma_f(k)$, $\sigma_\alpha - \sigma_f(k-1)$, $\sigma_f(k-1)$ and α^l and s_k. Now, we prove that (s_{k+1}, ϵ) satisfy the estimates (4.15) and the following:

$$\epsilon = \mu_w \Delta\sigma_{s_k} + K_w \alpha_1^l + O(1)x_0|\Delta\sigma_\alpha| + O(1)x_0|\Delta\sigma_{s_k}|. \tag{4.22}$$

Direct computation gives

$$\epsilon = \epsilon_1 + K_1'\big(\bar\sigma_f(k) - \bar\sigma_f(k-1)\big) + K_2'\big(\bar\sigma_\alpha - \bar\sigma_f(k) - \big(\sigma_\alpha - \sigma_f(k-1)\big)\big)$$

and

$$s_{k+1}=s'_{k+1}+K''_1\big(\bar\sigma_f(k)-\bar\sigma_f(k-1)\big)+K''_2\big(\bar\sigma_\alpha-\bar\sigma_f(k)-\big(\sigma_\alpha-\sigma_f(k-1)\big)\big),$$

where

$$\epsilon_1=\epsilon|_{\bar\sigma_\alpha-\bar\sigma_f(k)=\sigma_\alpha-\sigma_f(k-1)}$$

and

$$s'_{k+1}=s_{k+1}|_{\bar\sigma_\alpha-\bar\sigma_f(k)=\sigma_\alpha-\sigma_f(k-1)}$$

solve the following equations

$$\Phi\big(\alpha_1^l,0,h\big(\sigma_\alpha-\sigma_f(k-1),\sigma_f(k-1),G(s_k)\big)\big)=h\big(\sigma_\alpha-\sigma_f(k-1),\bar\sigma_f(k),\Phi_2\big(\epsilon_1,G\big(s'_{k+1}\big)\big)\big).$$

Now, we have the Taylor expansions

$$\epsilon_1=K_w\alpha_1^l+\epsilon_2$$

and

$$s'_{k+1}=K_s\alpha_1^l+s''_{k+1},$$

where $(\epsilon_2,s''_{k+1})=(\epsilon_1(\alpha_1^l=0),s'_{k+1}(\alpha_1^l=0))$ solves the following:

$$h\big(\sigma_\alpha-\sigma_f(k-1),\sigma_f(k-1),G(s_k)\big)=h\big(\sigma_\alpha-\sigma_f(k-1),\bar\sigma_f(k),\Phi_2\big(\epsilon_2,G\big(s''_{k+1}\big)\big)\big).\tag{4.23}$$

Note that ϵ_2 and s''_{k+1} are C^2-functions of σ_α, x_0 and s_k. Then, applying again Lemma 4.1 to ϵ_2 and s''_{k+1} gives

$$\epsilon_2=\widetilde K'_3x_0\big|\sigma_\alpha-\sigma_f(k-1)\big|=K'_3x_0|\Delta\sigma_\alpha|,$$
$$s''_{k+1}=s_k+K''_3x_0|\Delta\sigma_\alpha|,$$

where we use the following:

$$\epsilon_2|_{x_0=0}=\epsilon_2|_{\Delta\sigma_\alpha=0}=0$$

and

$$s''_{k+1}\big|_{x_0=0}=s''_{k+1}\big|_{\Delta\sigma_\alpha=0}=s_k.$$

Therefore, by Lemma 4.3 and by the above expansions of ϵ_j and s'_{k+1} and s''_{k+1}, we have the expansion for ϵ and s_k.

To compute the coefficients K_s and K_w, μ_s and μ_w, we differentiate Eq. (4.12) with respect to α^l and $\Delta\sigma_\alpha$, respectively, and take $\alpha^l=\Delta\sigma_\alpha=\Delta\sigma_{s_k}=0$ and $s_k=s_0$, then

$$r_1\big(G(s_0)\big)=K_wr_2\big(G(s_0)\big)+K_sG'_s(s_0)$$

and

$$\frac{\partial h(0,\sigma_f(k),G(s_0))}{\partial\Delta\sigma_{s_k}}=\mu_w r_2\big(G(s_0)\big)+\mu_s G'_s(s_0),$$

which give the formulas for K_s, K_w, μ_s and μ_w. Moreover,

$$\lim_{u_\infty\to+\infty}\mu_s=\frac{\cos\theta_0\sin\theta^0_{ma}}{\sin(\theta_0-\theta^0_{ma})}\in(-1,0),$$
$$\lim_{u_\infty\to+\infty}K_s=I\frac{\sin(2\theta^0_{ma})\cos^4(\theta_0+\theta_{ma})}{\sin(\theta^0_{ma}-\theta_0)\cos^2\theta_0}>0,$$

where

$$I=\lim_{u_\infty\to+\infty}\frac{1}{g(q)u_\infty}>0.$$

Here to get the above inequalities, we use the following:

$$\theta_0<0<\theta^0_{ma}$$

and

$$\theta_0\pm\theta^0_{ma}\in(-\pi/2,\pi/2).$$

The bounds of K'_j and K''_j, $j=2,3$, can be derived in a similar way.

Finally, we give the proof of (4.13) and (4.14). Denote the state between α^l and α^r as w^*. Thus by the construction of the approximate solution we have

$$\begin{aligned}
&\big\{w^-_\delta,w^*\big\}_1=O(1)\epsilon\Delta\sigma_\alpha,\\
&\big\{w^-_\delta,w^*\big\}_2=\epsilon+O(1)\epsilon\Delta\sigma_\alpha,\\
&\big\{w^*,w^+_\delta\big\}_j=\alpha^r_j+\gamma_j+O(1)\big(\Delta\big(\alpha^r,\gamma\big)+\Delta\sigma_\gamma|\gamma|+\Delta\sigma_\gamma|x_1|\big),
\end{aligned}$$

where the last estimate is obtained by the same way as in [16]. Then, combining these estimates with (4.22), we can get (4.13) and (4.14). The proof is complete. □

Lemma 4.5. *For Δx small,*

$$\big|\sigma_f(k-1)-s_k\big|\geqslant 4|\Delta\sigma_{s_k}|+O(1)x_0, \tag{4.24}$$

where the bound of $O(1)$ is independent of u_∞ and Δx.

Proof. We have $\sigma_f(k-1)=y_f(t_{k-1})/(t_{k-1}-x^*_1)$ and $\sigma_f(k)=y_f(t_k)/(t_k-x^*_2)$. Let $\sigma'_f(k)=y_f(t_k)/(t_k-x^*_1)$, then

$$\begin{aligned}
\left|s_k-\sigma_f(k-1)\right| &= \left|\frac{y_f(t_k)-y_f(t_k)}{t_k-t_{k-1}}-\sigma_f(k-1)\right| \\
&= \left|\frac{\sigma_f'(t_k-x_1^*)-\sigma_f(k-1)(t_{k-1}-x_1^*)}{t_k-t_{k-1}}-\sigma_f(k-1)\right| \\
&= \left|\frac{\sigma_f'(k)-\sigma_f(k-1)}{\Delta x}\big(t_k-x_1^*\big)\right| \\
&\geqslant 5\left|\sigma_f'(k)-\sigma_f(k-1)\right|
\end{aligned}$$

for small Δx. Moreover, direct computation gives

$$\left|\sigma_f'(k)-\sigma_f(k)\right| = \left|\sigma_f(k)x_0/\big(t_k-x_1^*\big)\right| = O(1)x_0.$$

Then, we have the desired result with the above estimates. The proof is complete. □

Denote $\theta_s(k)=|\sigma_f(k-1)-s_k|$, then,

Lemma 4.6. *For u_∞ large enough and Δx small enough, there hold the following*:

$$\theta_s(k)-\theta_s(k+1)\geqslant|\Delta\sigma_{s_k}|+K_s'\alpha_1^l+O(1)x_0|\Delta\sigma_\alpha|+O(1)x_0,\quad k\geqslant 0, \tag{4.25}$$

where $|K_s'|=|K_s|$ and K_s is given by Proposition 4.3; *the bound of the coefficient $O(1)$ of x_0 is independent of u_∞ and Δx.*

Proof. The proof is divided into several cases.

Case 1: $\sigma_f(k-1)<s_k<0$. Then, $\sigma_f(k)>\sigma_f(k-1)$ and $s_k>\sigma_f(k-1)$.

Moreover, if $s_{k+1}>\sigma_f(k)$, then, by Proposition 4.3,

$$\begin{aligned}
\theta_s(k)-\theta_s(k+1) &= s_k-\sigma_f(k-1)-\big(s_{k+1}-\sigma_f(k)\big) \\
&= (1-\mu_s)\Delta\sigma_{s_k}-K_s\alpha_1^l+O(1)x_0|\Delta\sigma_\alpha|+O(1)x_0|\Delta\sigma_{s_k}|;
\end{aligned}$$

if $s_{k+1}<\sigma_f(k)$, then, by Proposition 4.3 and Lemma 4.5,

$$\begin{aligned}
\theta_s(k)-\theta_s(k+1) &= s_k-\sigma_f(k-1)-\big(\sigma_f(k)-s_{k+1}\big) \\
&= 2\big(s_k-\sigma_f(k-1)\big)-(1-\mu_s)\Delta\sigma_{s_k}+K_s\alpha_1^l \\
&\quad +O(1)x_0|\Delta\sigma_\alpha|+O(1)x_0|\Delta\sigma_{s_k}| \\
&\geqslant (3+\mu_s)\Delta\sigma_{s_k}+K_s\alpha_1^l+O(1)x_0|\Delta\sigma_\alpha|+O(1)x_0|\Delta\sigma_{s_k}|+O(1)x_0.
\end{aligned}$$

Note that

$$\lim_{u_\infty\to\infty}\mu_s\in(-1,0),$$

then the above estimates give the proof for this case.

Case 2: $s_k < \sigma_f(k-1) < 0$. Then, $s_k < \sigma_f(k) < \sigma_f(k-1) < 0$, $\Delta\sigma_{s_k} < 0$. Therefore, if $s_{k+1} > \sigma_f(k)$, then, by Proposition 4.3 and Lemma 4.5,

$$\begin{aligned}\theta_s(k) - \theta_s(k+1) &= \sigma_f(k-1) - s_k - \big(s_{k+1} - \sigma_f(k)\big) \\ &= 2\big(\sigma_f(k-1) - s_k\big) - (1-\mu_s)\Delta\sigma_{s_k} + K_s\alpha_1^l \\ &\quad + O(1)x_0|\Delta\sigma_\alpha| + O(1)x_0|\Delta\sigma_{s_k}| \\ &\geqslant -(9+\mu_s)\Delta\sigma_{s_k} + K_s\alpha_1^l + O(1)x_0|\Delta\sigma_\alpha| \\ &\quad + O(1)x_0|\Delta\sigma_{s_k}| + O(1)x_0;\end{aligned}$$

if $s_{k+1} < \sigma_f(k)$, then, by Proposition 4.3,

$$\begin{aligned}\theta_s(k) - \theta_s(k+1) &= -s_k + \sigma_f(k-1) - \big(-s_{k+1} + \sigma_f(k)\big) \\ &= -(1-\mu_s)\Delta\sigma_{s_k} + K_s\alpha_1^l + O(1)x_0|\Delta\sigma_\alpha| + O(1)x_0|\Delta\sigma_{s_k}|.\end{aligned}$$

Note again that

$$\lim_{u_\infty\to\infty}\mu_s \in (-1, 0),$$

then the above estimates give the proof for this case. The proof is complete. $\square$

5. Glimm functional and the convergence of the approximate solutions

Let J be a space-like mesh curve connecting the mesh points. Then, we define the following:

Definition 5.1.

$$\begin{aligned}&L_0^k(J) = \sum\big\{|\alpha_k|\colon \alpha_k \text{ is the strength of the weak } k\text{-wave crossing } J\big\}, \quad k = 1, 2, \\ &L_1(J) = \theta_s(J), \\ &L_2(J) = \sum\big\{\big|\omega(A_k)\big|\colon A_k \in J^+\big\},\end{aligned}$$

and $L_c(J)$ is defined to be the total variation of $X^*|_J$ along the J. For any positive constant K_j, $j = 1, 2, 3, 4$, define

$$L(J) = L_0^1(J) + K_2L_0^2(J) + K_1L_1(J) + K_3L_2(J) + K_4L_c(J).$$

Remark 5.1. Different from the standard case, the linear part of Glimm functional includes three new terms L_1, L_2 and L_c, which play an important role in proving the monotonicity of Glimm functional as well as the quadratic part. The functional L_1 is initially introduced by Lien and Liu [16] which is to control the term $\mu_s\Delta\sigma_{s_k}$ in (4.15), while L_2 is introduced by Zhang [22] to control the term $K_A\omega$ in (4.4) and L_c is introduced in the present paper to control the term $O(1)x_0$ in (4.25).

Note that X^* is a piecewise constant function, $L_c(J)$ is well defined.

Definition 5.2.

$$\begin{aligned}
Q_0(J) &= \sum\bigl\{|\alpha||\beta|\colon \alpha \text{ and } \beta \text{ are the strengths of weak waves which are}\\
&\qquad \text{approaching and cross } J\bigr\},\\
Q_1(J) &= \sum\bigl\{|\alpha|(\sigma_\alpha - \sigma_*)\colon \alpha \text{ is a weak 1-wave crossing } J\bigr\},\\
Q_2(J) &= \sum\bigl\{|\alpha|(\sigma^* - \sigma_\alpha)\colon \alpha \text{ is a weak 2-wave crossing } J\bigr\},\\
Q_c(J) &= \sum Q_c^i(J),\\
Q_c^i(J) &= \bigl(X_0^i - X_0^{i-1}\bigr)\bigl(\sigma_c^i(J) - \sigma_*\bigr) \quad \bigl(X_0^0 = 0\bigr),\\
Q(J) &= Q_0(J) + Q_1(J) + Q_2(J) + Q_c(J).
\end{aligned}$$

Here σ_α denotes the σ-coordinate of the center for the wave α. $\sigma_c^i(J)$ is the σ-coordinate of the grid point where the center of the self-similar solutions passing through J changes from X_0^{i-1} to X_0^i. If the centers do not change anymore, then $Q_c(J) = 0$. Moreover, $\sigma^* = s_0 + \epsilon$, $\sigma_* = b_0 - M_1 \sum|\omega(A_k)|$, where s_0 is the velocity of the leading shock of the problem without perturbation. ϵ, M_1 are constants to be determined as in [6]. Note that ϵ and $\sum|\omega(A_k)|$ are chosen small, then the largeness of u_∞ implies smallness of $\sigma^* - \sigma_*$.

Based on the two definitions above, we define the Glimm functional as follows

$$F(J) = L(J) + K Q(J), \tag{5.1}$$

where K is big real number chosen to be large enough.

The following lemma is important to prove the monotonicity of Glimm scheme.

Lemma 5.1. *Let K_R, K_w, K_s and μ_w be given by Propositions 4.2 and 4.3. Then,*

$$\lim_{u_\infty\to+\infty}\bigl(K_R|K_w| + K_R|K_s||\mu_w|\bigr) < 1. \tag{5.2}$$

Proof. Lemmas 2.13, 2.15 and 4.2 give the following:

$$\begin{aligned}
\lim_{u_\infty\to+\infty} K_R|K_s||\mu_w| &= \frac{\cos^2(\theta_0+\theta_{ma}^0)}{\cos^2(\theta_0-\theta_{ma}^0)}\lim_{u_\infty\to+\infty}\left|\frac{\det(r_2,r_1)}{\det(r_2,G_s')}\right|\left|\frac{\det(\frac{\partial h}{\partial\Delta\sigma_{s_k}}, G_s')}{\det(r_2,G_s')}\right|\\
&= \frac{2\sin\theta_{ma}^0\cos\theta_{ma}^0|\sin\theta_0|\cos\theta_0}{\sin^2(\theta_0-\theta_{ma}^0)}
\end{aligned}$$

and

$$\lim_{u_\infty\to+\infty} K_R|K_w| = \left|\frac{\sin(\theta_0+\theta_{ma}^0)}{\sin(\theta_0-\theta_{ma}^0)}\right|. \tag{5.3}$$

Note that $\theta_0 \in (-\pi/2, 0)$ and $\theta_{ma}^0 \pm \theta_0 \in (-\pi/2, \pi/2)$, and that $\theta_{ma}^0 \in (0, \pi/2)$. If $\theta_{ma}^0 + \theta_0 < 0$, then

$$\lim_{u_\infty\to+\infty}\bigl(K_R|K_w| + K_R|K_s||\mu_w|\bigr) < \frac{2\sin\theta_{ma}^0\cos\theta_0 - \sin(\theta_{ma}^0+\theta_0)}{\sin(\theta_{ma}^0-\theta_0)} = 1;$$

if $\theta_{ma}^0 + \theta_0 > 0$, then

$$\lim_{u_\infty \to +\infty} \big(K_R|K_w| + K_R|K_s||\mu_w|\big) < \frac{2\cos\theta_{ma}^0|\sin\theta_0| + \sin(\theta_{ma}^0 + \theta_0)}{\sin(\theta_{ma}^0 - \theta_0)} = 1.$$

The proof is complete. □

This lemma leads to the following.

Lemma 5.2. *There exist positive constants K_1 and K_2 such that*

$$\lim_{u_\infty \to +\infty} \big(K_2|K_w| + K_1|K_s|\big) < 1,$$
$$\lim_{u_\infty \to +\infty} \big(K_2|\mu_w| - K_1\big) < 0,$$
$$\lim_{u_\infty \to +\infty} (K_2 - K_R) > 0.$$

Proof. Let $K_R^* = \lim_{u_\infty \to +\infty} K_R$, $K_w^* = \lim_{u_\infty \to +\infty} |K_w|$, $K_s^* = \lim_{u_\infty \to +\infty} |K_s|$ and $\mu_w^* = \lim_{u_\infty \to +\infty} |\mu_w|$. Then, by Lemma 5.1,

$$K_R^*\big(K_w^* + K_s^*\mu_w^*\big) < 1.$$

Therefore, we can choose some positive constant K_2 such that

$$K_2 > K_R^*, \qquad K_2\big(K_w^* + K_s^*\mu_w^*\big) < 1, \tag{5.4}$$

which leads to

$$K_2\mu_w^* K_s^* < 1 - K_2 K_w^*.$$

Then, we can choose a positive constant K_1 such that

$$K_1 > K_2\mu_w^*, \qquad K_1 K_s^* < 1 - K_2 K_w^*. \tag{5.5}$$

Now, the desired result comes from (5.4) and (5.5). The proof is complete. □

Let I_k be the mesh curve in the strip $\{(x, y) \mid t_{k-1} \leqslant x \leqslant t_k\}$ for any $k \geqslant 0$, that is, I_k is the space-like curve connecting all mesh points in the strip $\{(x, y) \mid t_{k-1} \leqslant x \leqslant t_k\}$.

Proposition 5.1. *Suppose that u_∞ is sufficiently large and that $\sigma^* - \sigma_*$ and $\sum |\omega(A_k)|$ are sufficiently small. There exist positive constants K and K_j, $j = 1, 2, 3, 4$, and δ independent of k such that if $F(I_k) < \delta$ then*

$$F(I_{k+1}) \leqslant F(I_k). \tag{5.6}$$

Proof. Choose u_∞ large enough so that

$$\big(K_2|K_w| + K_1|K_s|\big) < 1 - \eta_0,$$
$$\big(K_2|\mu_w| - K_1\big) < -\eta_0,$$
$$(K_2 - K_R) > \eta_0,$$

for some $\eta_0 > 0$ and for the constants K_1 and K_2 given in the above Lemma 5.2. Now we prove the result. The proof is carrying out by induction as in [16].

Let I and J be any pair of mesh curves with $I_k < I < J < I_{k+1}$ and Λ represents the region between I and J. We consider three special cases as in Section 4.

Case 1. Λ is the diamond covering part of the approximate boundary as in Section 4.2. Using the same notations as in Section 4.2, we have by direct computation

$$\begin{aligned}
&\Delta L_0^1 \leqslant K_R|\alpha| + K_A|\omega| + O(1)u_\infty x_0|\Delta\sigma|,\\
&\Delta L_0^2 = -|\alpha|,\\
&\Delta L_2 = -|\omega|,\\
&\Delta Q = O(1)L(I)\big(|\alpha| + |\omega|\big).
\end{aligned}$$

Here and in the sequel we denote that

$$\Delta L = L(J) - L(I), \qquad \Delta L_0^i = L_0^i(J) - L_0^i(I), \qquad \Delta L_c = L_c(J) - L_c(I), \qquad \Delta L = L(J) - L(I),$$

and

$$\Delta Q = Q(J) - Q(I), \qquad \Delta Q_i = Q_i(J) - Q_i(I), \qquad \Delta Q_c = Q_c(J) - Q_c(I).$$

Then,

$$\Delta L \leqslant (K_R - K_2)|\alpha| + (K_A - K_3)|\omega| + O(1)u_\infty x_0|\Delta\sigma|.$$

Since the perturbation of the boundary only occurs in a compact set, by Lemma 3.1 we have $|x_0| = O(1)|\omega|$. Therefore, using Lemma 5.2 and choosing large positive constants K_3 and K_4 and small $\delta > 0$, we can prove the result for this case.

Case 2. $\Lambda = \Delta_{k,j_f-1} \cup \Delta_{k,j_f} \cup \Delta_{k,j_f+1}$ as in Section 4.3, that is, Λ covers part of the leading shock front. Using the notations in Section 4.3, we compute that

$$\begin{aligned}
&\Delta L_0^1 = -\big|\alpha_1^l\big| + O(1)Q(\Lambda),\\
&\Delta L_0^2 \leqslant |\mu_w\Delta\sigma_{s_k}| + \big|K_w\alpha_1^l\big| + O(1)Q(\Lambda),\\
&\Delta L_1 \leqslant -|\Delta\sigma_{s_k}| + \big|K_s\alpha_1^l\big| + O(1)x_0|\Delta\sigma_\alpha| + O(1)x_0,\\
&\Delta L_c = -|x_0|.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\Delta L &\leqslant -\big|\alpha_1^l\big| + O(1)Q(\Lambda) + K_2|\mu_w\Delta\sigma_{s_k}| + K_2|K_w|\big|\alpha_1^l\big| - K_1|\Delta\sigma_{s_k}|\\
&\quad + K_1\big|K_s\alpha_1^l\big| + O(1)x_0|\Delta\sigma_\alpha| + O(1)x_0 - K_4|x_0|\\
&= \big|\alpha_1^l\big|\big(-1 + K_2|K_w| + K_1|K_s| + O(1)\Delta\sigma_\alpha\big) + \Delta\sigma_{s_k}\big(K_2|\mu_w| - K_1 + O(1)\Delta\sigma_\alpha\big)\\
&\quad + |x_0|\big(-K_4 + O(1) + O(1)\Delta\sigma_\alpha\big) + O(1)\big(\Delta\big(\alpha^r,\gamma\big) + \Delta\sigma_\gamma\big(x_1 + |\gamma|\big)\big)\\
&\leqslant \big|\alpha_1^l\big|\big(-\eta_0 + O(1)\Delta\sigma_\alpha\big) + |\Delta\sigma_{s_k}|\big(-\eta_0 + O(1)\Delta\sigma_\alpha\big) + |x_0|\big(-K_4 + O(1)\big)\\
&\quad + O(1)\big(\Delta\big(\alpha^r,\gamma\big) + \Delta\sigma_\gamma\big(x_1 + |\gamma_1|\big)\big).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Delta Q_0 \quad &\leqslant \big(|\mu_w \Delta\sigma_{s_k}| + \big|K_w \alpha_1^l\big| - \big|\alpha_1^l\big|\big)L(I) + O(1)Q(\Lambda)L(I) - \Delta(\alpha,\gamma),\\
\Delta Q_1 + \Delta Q_2 &\leqslant -|\gamma_1||\Delta\sigma_\gamma| - \big|\alpha_1^l \Delta\sigma_\alpha\big| + O(1)Q(\Lambda)(\sigma^* - \sigma_*)\\
&\quad + \big(|\mu_w \Delta\sigma_{s_k}| + \big|K_w \alpha_1^l\big|\big)(\sigma^* - \sigma_*),\\
\Delta Q_c(J) \quad &= -|x_1||\Delta\sigma_\gamma|.
\end{aligned}$$

Thus, using Lemma 5.2 and by choosing large K_3, K_4 and K, we get

$$\Delta L + K\Delta Q \leqslant 0$$

for small $\sigma^* - \sigma_*$ and for $F(I) \ll 1$. This leads to the monotonicity of the Glimm functional F.

Case 3. Λ lies between the approximate boundary and the leading shock front. The proof for this case is the same as in [16].

The proof is complete. □

Proposition 5.1 implies that the total variation of the approximate solution is uniformly bounded. Then in the standard way as in [6,16], we can prove the main theorem of our paper as follows.

Theorem 5.1. *Suppose that the conditions* (A1) *and* (A2) *are satisfied, and that* $1 < \gamma < 3$ *and* $-b_* < b_0 < 0$. *If* $T.V.b'(\cdot)$ *is sufficiently small and* u_∞ *is sufficiently large, then problem* (1.1)–(1.4) *has a global solution* $U(x, y)$ *with bounded total variation. The solution contains a* 1*-shock front, which is a small perturbation of* $y = s_0 x$*, and the solution in between the shock front and surface of the cone is a small perturbation of the self-similar solution of the problem* (2.14)–(2.18)*.*

Remark 5.2. In this theorem, the definition of b_* can be founded in (2.23), and s_0 denote the location of shock front when the location of the surface of the cone is given by $y = b_0 x$.

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